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Modes of finite-amplitude three-dimensional convection in rectangular boxes of fluid-saturated porous material

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Steady, three-dimensional convection in rectangular boxes of fluid-saturated porous material with square horizontal cross-section heated from below is found to be non-unique. The properties of a special class of solutions exhibiting a high degree of symmetry are determined as a function of box size and Rayleigh number. The stability of these solutions to general three-dimensional perturbations is also determined. In some cases, when these solutions are found to be unstable, the alternative forms of three-dimensional convection are presented. Multiple three-dimensional steady states are given for a few particular values of box size and Rayleigh number.

1. Introduction

This paper is part of a continuing effort (Straus & Schubert 1978, 1979; Schubert & Straus 1979) to establish the fundamental properties of finite-amplitude three-dimensional thermal convection in fluid-saturated rectangular boxes of porous material heated from below. Our previous studies, as well as those of Holst & Aziz (1972) and Horne (1979), have concentrated on flows in cubes. According to linear theory, steady three-dimensional convection in a cube can occur if the Rayleigh number R exceeds $4\pi^2$ (Beck 1972). For R in the range $4\pi^2$–$4{\cdot}5\pi^2$, the motion is a superposition of orthogonal two-dimensional rolls (Zebib & Kassoy 1978). When R exceeds $4{\cdot}5\pi^2$ fully three-dimensional modes of convection can occur. The convective mode which becomes unstable at $R = 4{\cdot}5\pi^2$ according to linear theory, the so-called (1, 1, 1) mode, has ascending flow at diagonally opposite vertical edges of the cube and descending flow at the other diagonally opposed edges, as shown in figure 1. This three-dimensional convective mode is a member of a class of solutions possessing the following properties: invariance of the flow to an interchange of horizontal co-ordinates; symmetry of the temperature field about diagonals of horizontal planes; antisymmetry of the temperature field with respect to reflexion about the horizontal midplane; and rotation of 90° about a vertical axis through the centre of the cube (Straus & Schubert 1979). All the steady three-dimensional solutions that we reported in our earlier papers have possessed these characteristics; henceforth, we refer to such solutions as symmetric modes of convection.

The symmetric solutions form a closed subset of all possible convective motions. The subset does not include the combination of orthogonal two-dimensional rolls which

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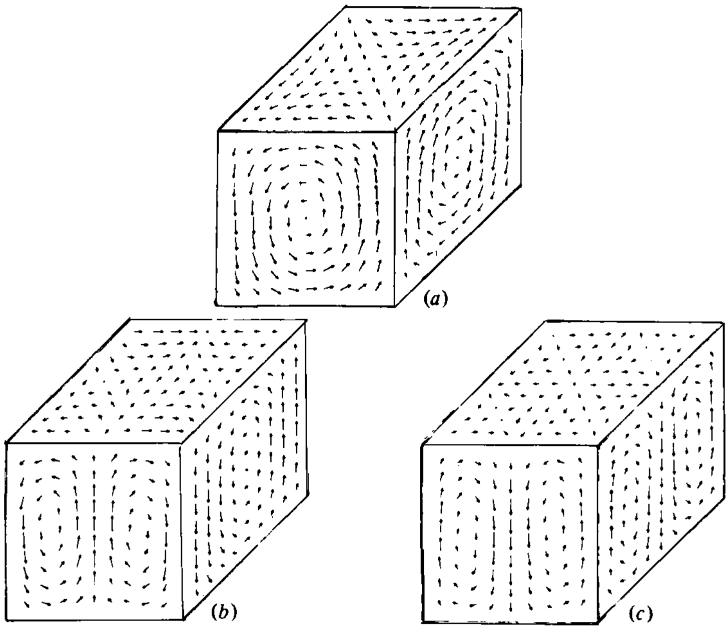

FIGURE 1. Alternative forms of convection in a cube: (a) (1, 1, 1) mode; (b) (1, 1, 2) mode; (c) (1, 2, 2) mode. The length of an arrow is proportional to the velocity at the location of the base of the arrow.

exists for $R > 4\pi^2$, for example. Steady three-dimensional symmetric convective motions in a cube have been calculated for R in the range $4{\cdot}5\pi^2$ to 150 by Straus & Schubert (1979). Schubert & Straus (1979) found that such solutions exist for R as large as 300, but that symmetric three-dimensional convection becomes oscillatory for R in excess of a value between 300 and 320 (see also Horne 1979). The steady symmetric three-dimensional convective motions in a cube are dominated by the (1, 1, 1) mode at all Rayleigh numbers between $4{\cdot}5\pi^2$ and 300.

Linear stability theory predicts that a number of nonsymmetric two-dimensional and fully three-dimensional modes of convection in the cube are possible at Rayleigh numbers less than 300. Thus one would expect that steady three-dimensional convection in a cube would be non-unique for $R > 4\pi^2$; non-symmetric forms of convection should exist at the same value of R for which we have obtained symmetric solutions. In fact, the question of the stability of the symmetric solutions to nonsymmetric perturbations immediately arises because the symmetric forms of convection constitute a subset of all possible convective modes. Accordingly, we have analysed the stability of the steady three-dimensional symmetric solutions in a cube to general (non-symmetric) perturbations. The results, discussed in detail later, show that symmetric forms of convection in a cube are stable for R less than some value between 200 and 220. For larger values of R, we have found that there are steady three-dimensional modes of convection in the cube which are dominated by the (1, 1, 1) mode and have Nusselt numbers Nu very close to those of the symmetric solutions; these solutions contain relatively insignificant contributions from nonsymmetric modes. These nonsymmetric (1, 1, 1)-dominated solutions in the cube become oscillatory for R in excess of some value between 300 and 350. Thus their properties are very similar to those of the symmetric solutions. This is not to say that non-symmetric modes of convection, quite unlike the symmetric solutions, do not also exist. We will present, as an example of the non-uniqueness of the solutions, a solution in the cubic geometry dominated by orthogonal two-dimensional rolls at a value of R for which symmetric fully three-dimensional convection is stable.

We have also extended our study of three-dimensional convection to non-cubic geometries. We consider rectangular boxes with square horizontal cross-section and varying height-to-width ratios. We have thus been able to delineate a region of stability for symmetric steady solutions in a Rayleigh-number–box-size plane. For a few box sizes and values of R at which symmetric convection is unstable, we have calculated the forms of steady non-symmetric motion which exist instead.

2. Linear stability theory

The critical Rayleigh number R_c for the onset of convection in a rectangular box of fluid-saturated porous material heated from below with height d and horizontal dimensions l and b is (Beck 1972)

$$R_c = \frac{\pi^2\{n^2+j^2(d/l)^2+m^2(d/b)^2\}^2}{j^2(d/l)^2+m^2(d/b)^2}, \tag{1}$$

where $n, j, m = 0, 1, 2, \ldots$ are integers specifying the vertical (z) and horizontal (x, y) structures of the Fourier components of the temperature and motion fields. The disturbance temperature field, for example, is a superposition of modes of the form

$$\sin(n\pi z/d)\cos(j\pi x/l)\cos(m\pi y/b).$$

In this paper we only consider boxes with square horizontal cross-section $l = b$; the critical Rayleigh numbers in this circumstance are given by

$$R_c = \frac{\pi^2\{n^2+(d/l)^2(j^2+m^2)\}^2}{(d/l)^2(j^2+m^2)}. \tag{2}$$

Convection can occur if the Rayleigh number R exceeds the minimum value of R_c; all modes with $R_c \leqslant R$ are convectively unstable. The Rayleigh number is

$$R = \alpha g \rho^2 K c\, d\, \Delta T/\mu\kappa, \tag{3}$$

where α is the coefficient of thermal expansion of the fluid, g is the acceleration of gravity, ρ is the fluid density, K is the permeability of the porous medium, c is the fluid specific heat, ΔT is the temperature difference between the lower and upper isothermal surface of the box, μ is the fluid viscosity and κ is the average thermal conductivity of the fluid and solid matrix.

Critical Rayleigh numbers for some of the simpler modes of convection are shown as functions of height-to-width ratio in figure 2. The figure includes two-dimensional modes (dotted curves), and fully three-dimensional symmetric (solid curves) and non-symmetric (dashed curves) modes. Throughout this paper we adopt the convention of labelling convective modes in the order (n, j, m). Clearly, for a given box size and values of R which are not too small, there are many two-dimensional and fully three-dimensional non-symmetric modes which can contribute to a general convective motion. The modes included in figure 2 are only a few of the very large number of modes which can exist at moderate supercritical Rayleigh numbers, especially in boxes shorter than

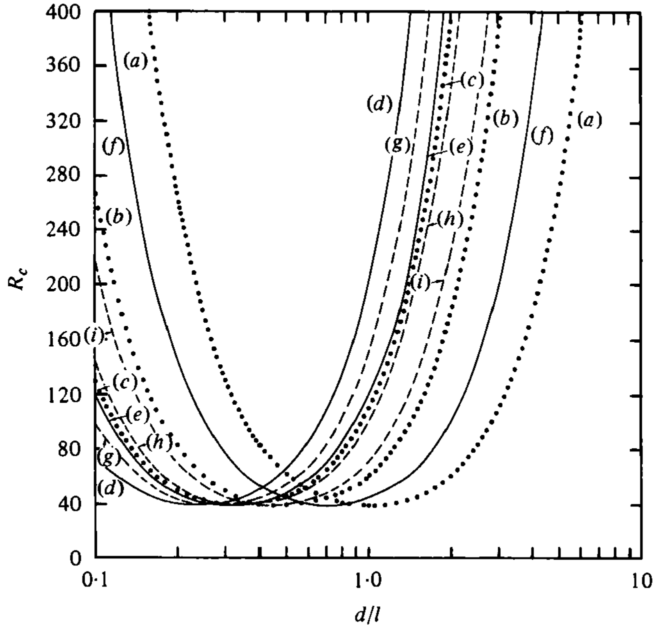

FIGURE 2. Critical Rayleigh numbers R_c for the onset of convection as functions of box height-to-width ratio d/l for a number of two- and three-dimensional modes of convection. Dotted curves are two-dimensional modes, dashed curves are non-symmetric three-dimensional modes, and solid curves are symmetric three-dimensional modes. (*a*) (1, 0, 1), (1, 1, 0); (*b*) (1, 0, 2), (1, 2, 0); (*c*) (1, 0, 3), (1, 3, 0); (*d*) (1, 3, 3); (*e*) (1, 1, 3), (1, 3, 1); (*f*) (1, 1, 1); (*g*) (1, 3, 2), (1, 2, 3); (*h*) (1, 2, 2); (*i*) (1, 2, 1), (1, 1, 2).

they are wide. We have specifically identified these particular modes here because they are the simplest of their respective types, and some of them dominate the finite-amplitude motions we discuss later.

The modes labelled (1, 0, 1), (1, 1, 0), (1, 0, 2), (1, 2, 0), (1, 0, 3), (1, 3, 0) are two-dimensional rolls with horizontal axes parallel to the sides of the box. The modes (1, 0, 1) and (1, 1, 0) are single rolls, while (1, 0, 2), (1, 2, 0) and (1, 0, 3), (1, 3, 0) have two and three cells in the horizontal, respectively. The other modes in figure 2 are fully three-dimensional. Modes (1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3) are symmetric, while (1, 2, 2), (1, 1, 2), (1, 2, 1), (1, 2, 3), (1, 3, 2) are non-symmetric. Figure 1 illustrates the flow patterns of the (1, 1, 1), (1, 1, 2) and (1, 2, 2) modes. All other modes can be easily visualized as extensions of the patterns in figure 1.

Since there are, in general, many two-dimensional and non-symmetric three-dimensional modes that can occur in a rectangular box for a given R, one would expect that the symmetric steady states we have previously reported for $4{\cdot}5\pi^2 < R \leqslant 300$ are non-unique. Also, it is possible that under certain circumstances these symmetric solutions may be unstable to the growth of general perturbations and not be realizable in a fully three-dimensional situation. In what follows, we first extend our previous calculations of symmetric convective states to the rectangular boxes with $d/l \neq 1$. We then delineate the region of stability of symmetric three-dimensional steady states, in the R *vs.* d/l plane. We also determine the form of steady three-dimensional convection in a few cases when the symmetric solutions are not stable. Finally, we

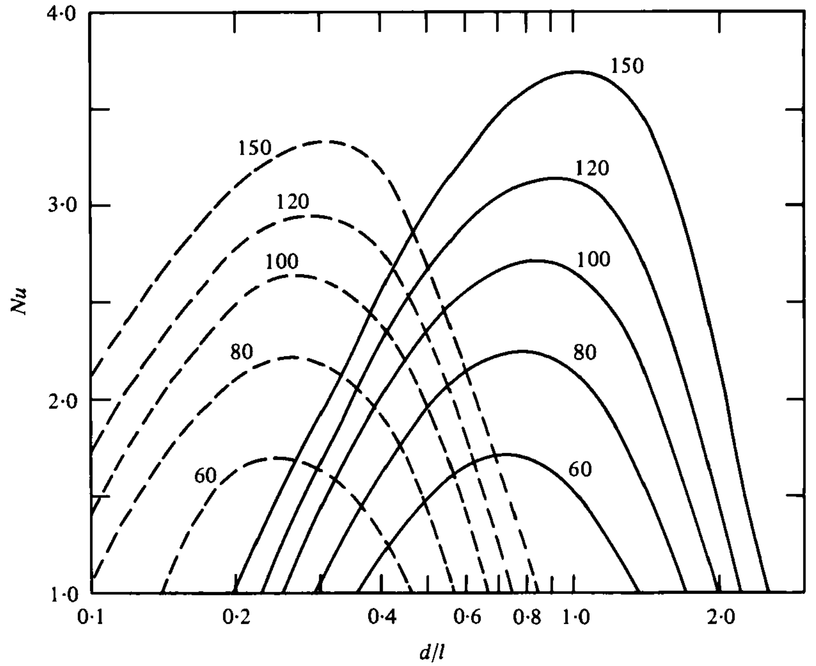

FIGURE 3. Nusselt number *vs.* d/l with R as a parameter for symmetric forms of convection. Solid curves refer to states dominated by the (1, 1, 1) mode. Dashed curves are for solutions dominated by the (1, 3, 3) mode.

illustrate the non-uniqueness of steady three-dimensional convection by obtaining a non-symmetric solution for the cube at a value of R for which a stable symmetric solution also exists.

3. Symmetric steady states in non-cubic boxes

The steady, finite-amplitude, symmetric, three-dimensional convective solutions in a cube for $4{\cdot}5\pi^2 < R \leqslant 300$ have already been described in our earlier papers. We used the Galerkin procedure described in those papers to produce symmetric steady states in non-cubic boxes. The solutions reported here were obtained with $N = 10$, where the parameter N is a measure of the number of Fourier coefficients used in a calculation. The truncation criterion is $n+j+m \leqslant N$. We have not attempted to compute all possible symmetric solutions at each value of R and d/l.

Figures 3 and 4 summarize the Nusselt numbers of these solutions. The Nusselt number Nu is the ratio of the average vertical heat flux to the heat flux in the basic conduction state. The solid curves in figure 3 refer to states dominated by the (1, 1, 1) mode; the dashed curves are for (1, 3, 3)-dominated convection. The Nusselt numbers in figure 4 are for symmetric forms of convection dominated by (1, 3, 1) and (1, 1, 3) modes. For convection dominated by a particular mode, and for a given d/l, Nu increases as R increases. As a function of d/l for a given R, Nu exhibits a maximum in the range of d/l for which solutions are possible. The maxima occur at values of d/l near those that minimize R_c according to linear theory. At a given R, the type of symmetric solution which transports the most heat depends on d/l. As an example, at

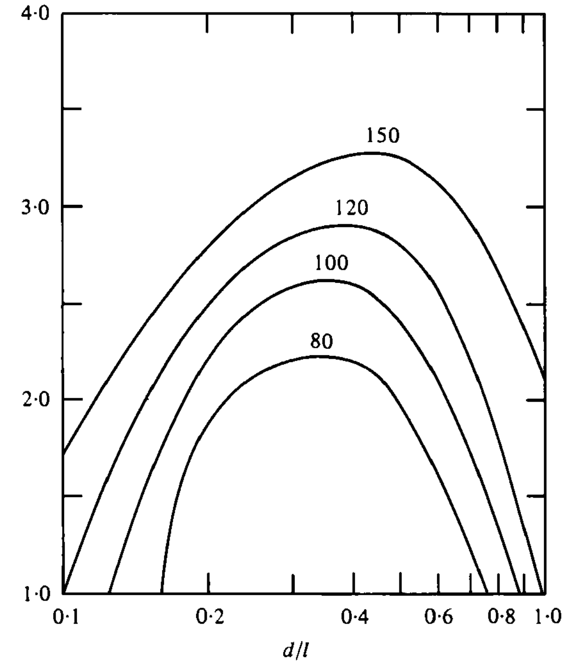

FIGURE 4. Same as figure 3 for (1, 3, 1)- and (1, 1, 3)-dominated symmetric solutions.

$R = 150$, symmetric convection dominated by the $(1, 1, 1)$ mode transfers more heat than does $(1, 3, 1)$-dominated convection for $d/l \gtrsim 0{\cdot}57$. For this same value of R, $(1, 3, 1)$-dominated convection transfers more heat than either $(1, 1, 1)$- or $(1, 3, 3)$-dominated convection when d/l lies between about 0·38 and 0·57. For $d/l \lesssim 0{\cdot}38$, the $(1, 3, 3)$-dominated state has a Nusselt number larger than the values of Nu for the other two types of convection.

4. Stability of the symmetric solutions

With the Galerkin technique, a solution to the equations of motion and temperature is obtained by solving a set of coupled, nonlinear, first-order, ordinary differential equations for the time behaviour of the Fourier expansion coefficients ϕ_{njm} (see, for example, Straus & Schubert 1979). The stability of a steady state specified by a set of values $\bar{\phi}_{njm}$ can be determined by perturbing the steady state with quantities ϕ'_{njm} and determining whether the perturbations grow or decay with time. The linearized set of equations for the temporal evolution of ϕ'_{njm} is of the form

$$\dot{\phi}'_{njm} = M_{njmn'j'm'}\phi'_{n'j'm'}, \tag{4}$$

where the dot indicates differentiation with respect to time. The elements of $M_{njmn'j'm'}$ depend on the steady-state coefficients $\bar{\phi}_{njm}$. The eigenvalues of $M_{njmn'j'm'}$ determine whether the steady state is stable or not; if there are no eigenvalues with positive real part then the steady state is stable. When calculating these eigenvalues for the symmetric steady states, we found that the eigenvalue with the largest real part was always real.

The elements of $M_{njmn'j'm'}$ were calculated numerically by initializing the ϕ_{njm} with

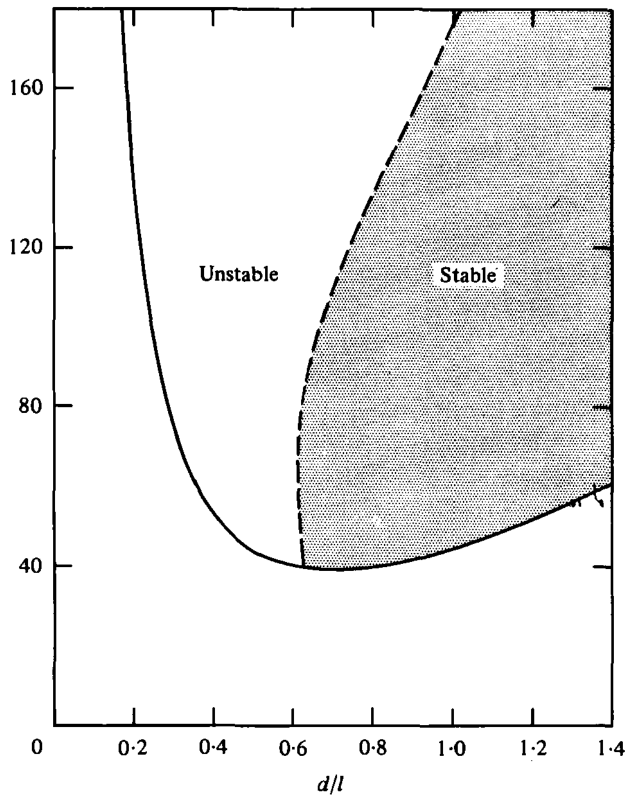

FIGURE 5. Region of stability of (1, 1, 1)-dominated symmetric convection in a R *vs.* d/l plane.

the values $\bar{\phi}_{njm}$ for the symmetric steady state. Coefficients ϕ_{njm} not allowed by the symmetry requirements of the steady state were initialized to zero. The system was then perturbed by adding a small non-zero quantity to a particular ϕ_{njm}. All quantities $\dot{\phi}_{njm}$ were then calculated; the ratio of a particular $\dot{\phi}_{njm}$ to the perturbation amplitude of some other coefficient $\phi_{n'j'm'}$ determines an element of $M_{njmn'j'm'}$. Perturbations of all coefficients satisfying the truncation criterion $n+j+m \leqslant N$ were allowed. (Recall that all the symmetric steady states were calculated for $N = 10$.) The value $N = 10$ is sufficiently large to include the interactions between the symmetric modes (1, 1, 1), (1, 3, 1), (1, 1, 3), (1, 3, 3) and the non-symmetric ones (1, 0, 1), (1, 1, 0), (1, 0, 2), (1, 2, 0), (1, 0, 3), (1, 3, 0), (1, 1, 2), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 2) in the stability analysis.

The region of R *vs.* d/l parameter space in which (1, 1, 1)-dominated symmetric solutions are stable is shown in figure 5. The solid curve is the linear stability boundary for the onset of convection in the (1, 1, 1) mode. The dashed curve is the boundary, determined from the analysis described just above, separating symmetric (1, 1, 1)-dominated steady solutions which are stable to general three-dimensional perturbations from those that are not. Symmetric (1, 1, 1)-dominated steady solutions in a cube, for example, are stable for $4{\cdot}5\pi^2 \leqslant R < 180$. Actually, these solutions are stable for even larger R because the location of the dashed stability boundary depends somewhat on N. To test the sensitivity of the stability boundary to the value of N used in its

R	Symmetric	Non-symmetric
200	4·41	4·42
250	5·00	4·95
300	5·43	5·30

TABLE 1. A comparison of Nusselt numbers for symmetric and non-symmetric steady convection in a cube ($N = 10$).

computation, we carried out stability calculations for symmetric (1, 1, 1) steady states in a cube with $N = 12$ and found that these solutions are stable for R less than some value between 200 and 220. Thus, the stability boundary of figure 5 is accurate to about 10 % at the larger values of R treated, but the accuracy improves with decreasing R. Figure 5 shows that, in a fully three-dimensional situation, symmetric (1, 1, 1)-dominated steady solutions are not realizable in boxes with $d/l \lesssim 0{\cdot}6$.

We also tested the stability of symmetric (1, 3, 3)- or (1, 3, 1)-, (1, 1, 3)-dominated steady solutions and found them all to be unstable to general three-dimensional perturbations. Thus none of these solutions is realizable in a fully three-dimensional situation.

5. Non-symmetric steady states

What is the nature of the non-symmetric forms of convection when d/l and R are such that symmetric solutions are unstable? Although we have not attempted to answer this question in a systematic and detailed manner, we have calculated a number of non-symmetric steady states for several values of R and d/l. We first consider the cubic geometry. According to figure 5, symmetric solutions are unstable for $R > 180$ when $d/l = 1$ and $N = 10$. We have computed non-symmetric steady states with $R = 200$, 250 and 300 when $d/l = 1$ and $N = 10$. These solutions are all essentially similar to the symmetric steady states we reported previously, in the sense that they are dominated by the symmetric (1, 1, 1) mode with only small contributions from non-symmetric modes. Table 1 shows how close the Nusselt numbers are for the symmetric and nonsymmetric solutions. While these particular non-symmetric solutions are very much like the symmetric ones, there are likely to be other non-symmetric solutions which are quite distinct.

Non-symmetric convection in a cube at $R = 350$ was found to be oscillatory with $N = 10$. Thus our earlier conclusion (Schubert & Straus 1979) that three-dimensional convection in a cube becomes oscillatory when R lies somewhere between 300 and 320 is essentially unaltered by the inclusion of the non-symmetric modes (as least for those non-symmetric solutions dominated by the (1, 1, 1) symmetric mode).

As one example of the fundamental non-uniqueness of three-dimensional convection, we have obtained a non-symmetric steady solution in a cube dominated by the orthogonal rolls (1, 1, 0), (1, 0, 1) at a Rayleigh number of 80 (the solution was calculated with $N = 8$). The Nusselt number for this solution is 2·24. The symmetric (1, 1, 1)-dominated solution is stable at this value of R and has a Nusselt number of 2·16 ($N = 8$). Thus, initial conditions determine which of these stable steady forms of three-dimensional convection in a cube would be realized at $R = 80$.

We have also obtained some non-symmetric steady solutions for the case $d/l = 0{\cdot}5$

at $R = 80$ and 100. For these values of R and d/l there are no stable symmetric steady states. At $R = 80$, the non-symmetric solution is mainly (1, 2, 1) and (1, 1, 2), and $Nu = 2{\cdot}23$ ($N = 8$). At $R = 100$ we calculated two solutions, one dominated by (1, 2, 1) and (1, 1, 2), and the other a mixture of many modes including (1, 0, 1), (1, 1, 0), (1, 0, 2), (1, 2, 0), (1, 1, 2) and (1, 2, 1). Nu for these solutions are 2·61 and 2·67 ($N = 8$), respectively. Again, initial conditions must determine which of these solutions is realized.

6. Summary and discussion

We have extended our studies of three-dimensional convection in rectangular boxes of fluid-saturated porous material by computing the properties of steady symmetric solutions in boxes with square horizontal cross-section. This subset of all possible solutions possesses special symmetry properties making its computation much easier (in terms of computer time and storage requirements) than that of a general solution. However, symmetric solutions are not always stable to general three-dimensional perturbations. Accordingly, we have delineated the stability boundary of the symmetric solutions in a Rayleigh-number–box-size plane.

In a number of cases when symmetric solutions are unstable, we have calculated the alternative forms of steady three-dimensional convection. In cubes, symmetric convection is unstable for sufficiently high Rayleigh number; in this circumstance, we have found non-symmetric steady states whose characteristics are quite similar to those of the unstable symmetric solutions. In rectangular boxes which are much wider than they are tall, symmetric solutions are generally found to be unstable. In these cases we have found non-symmetric steady states which are quite distinct from their unstable symmetric counterparts.

Steady three-dimensional convection in rectangular boxes is highly non-unique. As examples of this we have presented a non-symmetric solution and a stable symmetric one at the same value of R in a cube, and two non-symmetric states in a short rectangular box at the same value of R. These multiple solutions at identical values of d/l and R have quite distinct characteristics.

More work is clearly needed to describe the features of steady non-symmetric convection, including their forms and heat transfer capabilities. If this could be done in some systematic way then we would at least have a qualitative understanding of the extent of the non-uniqueness in solutions.

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